③ Q.No. → Prove that the two circles which pass through the points $(0, a)$ and $(0, -a)$ and touch the line $y = mx + c$ will cut orthogonally if $c^2 = a^2(2 + m^2)$.

Ans. → Let $x^2 + y^2 + 2gx + 2fy + c = 0$ ① be the equation of circle.

As point passes through $(0, a)$

$$a^2 + 2fa + c = 0 \quad —②$$

As point passes through $(0, -a)$;

$$0 + a^2 + 0 - 2fa + c = 0$$

$$\text{or, } a^2 - 2fa + c = 0 \quad —③$$

from ② − ③

$$4fa = 0$$

$$\therefore f = \frac{0}{4a} = 0$$

putting $f = 0$ in ②

$$a^2 + c = 0$$

$$\therefore c = -a^2$$

Hence, the eqn. of the circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$x^2 + y^2 + 2gx + 0 - a^2 = 0$$

$$x^2 + y^2 + 2gx - a^2 = 0$$

Centre of the circle

$$(-g, 0)$$

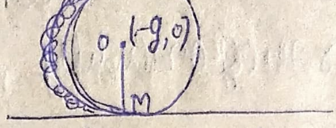
and radius $= \sqrt{g^2 + 0 + a^2}$

Hence $\perp$ distance from the centre $(-g, 0)$ of the line,

$y = mx + c$ i.e. $mx - y + c = 0$

is equal to radius.

$$\frac{-mg + 0 + c}{\sqrt{m^2 + 1^2}} = \sqrt{g^2 + a^2}$$



$mx - y + c = 0$

$$\frac{c - mg}{\sqrt{m^2 + 1}} = \sqrt{g^2 + a^2}$$

Squaring both sides, we have

$$(c - gm)^2 = (m^2 + 1)(g^2 + a^2)$$

or, $\cancel{g^2 m^2} - 2gmc + c^2 = \cancel{m^2 g^2} + m^2 a^2 + g^2 + a^2$

$g^2 + 2gmc + a^2 + m^2 a^2 - c^2 = 0$

or, $g^2 + 2gmc + a^2(1 + m^2) - c^2 = 0$

It is quadratic in g, (4)

Let g_1 and g_2 be the roots of the above eqn (4)

$g_1 + g_2$

and, $g_1 \cdot g_2$

eqn. $= g^2 + 2gmc + a^2(1 + m^2) - c^2 = 0$

$$\therefore g_1 + g_2 = -\frac{b}{a}$$

$$= -\frac{2mc}{1}$$

$$g_1 \cdot g_2 = \frac{c}{a}$$

$$= \frac{a^2(1 + m^2) - c^2}{1}$$

Hence, the eqn. of two circle be

$$x^2+y^2+2g_1x-a^2=0$$

and $x^2+y^2+2g_2x-a^2=0$

When these two circles cut orthogonally the, $2g_1g_2+2f_1f_2=C_1+C_2$

$$\text{or, } 2\,\frac{a^2(1+m^2)-c^2}{1}+0=-a^2-a^2$$

$$\cancel{2}\left[a^2(1+m^2)-c^2\right]=-\cancel{2}a^2$$

$$a^2(1+m^2)+a^2=c^2$$

$$a^2(1+m^2-1)=c^2$$

$$a^2(2+m^2)=c^2 \quad \underline{\text{proved}}$$

(5) QN.→ Find the equation of the circle orthogonal to the circles $x^2+y^2+3x-5y+6=0$ and $4x^2+4y^2-28x+29=0$ and whose centre lines on the line $3x+4y+1=0$.

~~Ans.→ Let $x^2+y^2+2gx+2fy+c=0$ be the eqn. of the circle orthogonal~~

Ans.→ ∵ The eqn. of the orthogonal circles

$$x^2+y^2+3x-5y+6=0 \quad \text{--- (1)}$$

and $4x^2+4y^2-28x+29=0$ --- (2)

Let $x^2+y^2+2gx+2fy+C=0$ be the eqn. of the circle orthogonal to (1) and (2)

$$\therefore \cancel{2}g\times\frac{3}{\cancel{2}}+\cancel{2}f\times-\frac{5}{\cancel{2}}=C+6$$

$$3g-5f=C+6 \quad \text{--- (3)}$$

from

$4x^2+4y^2-28x+29=0$

or, $x^2+y^2-7x+\frac{29}{4}=0$

As $x^2+y^2+2gx+2fy+c=0$ be orthogonal

to $x^2+y^2-7x+\frac{29}{4}=0$

$2g \times -\frac{7}{2} + 2f \times 0 = c+\frac{29}{4}$

or, $-7g = c+\frac{29}{4}$ — (4)

$\therefore c = -\frac{29}{4}$

As the centre of $x^2+y^2+2gx+2fy+c=0$

lines on the line $3x+4y+1=0$

$\therefore$ Centre of $x^2+y^2+2gx+2fy+c=0$

is $(-g,-f)$

$-3\times g+4\times -f+1=0$

or, $-3g-4f+1=0$

or, $3g+4f-1=0$

$3g+4f=1$ — (5)

$f=\frac{1}{4}$

(3) − (4)

$10g-5f=6-\frac{29}{4}$

$10g-5f=-\frac{5}{4}$

$2g-f=-\frac{1}{4}$

or, $8g-4f=-1$ — (6)

(5) + (6)

$11g=0$ $\therefore g=0$

putting the values of g, f and c in eqn.
~~①, we have~~ $x^2+y^2+2gx+2fy+c=0$, we have

$$x^2+y^2+0+2\times\frac{1}{4}y-\frac{29}{4}=0$$

or, $4x^2+4y^2+2y-29=0$

required equation.

⑥ Q.N. → obtain the cartesian equation of the circle which cuts orthogonally each of the three circles,

$x^2+y^2=16$, $x^2+y^2-14x+40=0$, ~~x^2+y^2-12y~~
$x^2+y^2-12y+32=0$.

Ans. → ~~∴ The three circles is~~

Let $x^2+y^2+2gx+2fy+c=0$ be the eqn. of the required circle

since the circle, $x^2+y^2+2gx+2fy+c=0$ cuts orthogonally the three circles,

$x^2+y^2=16$, $x^2+y^2-14x+40=0$

and $x^2+y^2-12y+32=0$

$\therefore 2g\times 0+2f\times 0=c-16$ — ①

$\therefore c=16$

$\cancel{2}g\times -\frac{14}{\cancel{2}}+2f\times 0=c+40$

or, $-14g=c+40$ — ②

and, $2g\times 0+\cancel{2}f\times -\frac{12}{\cancel{2}}=c+32$

or, $-12f=c+32$ — ③

Putting values of c in (2) and (3), we have

$-14g = 16+40$

or, $-14g = 56$

$\therefore g = -\frac{\cancel{56}\,4}{\cancel{14}\,2} = -4$

and $-12f = 16+32$

or, $-12f = 48$

$\therefore f = -\frac{\cancel{48}\,4}{\cancel{12}} = -4$

Putting the values of g, f and c in eqn.

$x^2+y^2+2gx+2fy+c=0$, we have.

$x^2+y^2+2x-4\cdot x+2x-4\cdot y+16=0$

or, $x^2+y^2-8x-8y+16=0$

required equation

(7) obtain the Cartesian equation of the circle which cuts orthogonally each of the three circles

$x^2+y^2+2x+17y+4=0$

$x^2+y^2+7x+6y+11=0$

and $x^2+y^2-x+22y+3=0$

Ans.→ Let $x^2+y^2+2gx+2fy+c=0$ be the eqn. of the required circle

Since the circle, $x^2+y^2+2gx+2fy+c=0$ cuts orthogonally the tree circles

$$x^2+y^2+2x+17y+4=0$$

$$x^2+y^2+7x+6y+11=0$$

and $$x^2+y^2-x+22y+3=0$$

$$\therefore \cancel{2}g \times \frac{2}{\cancel{2}} + \cancel{2}f \times \frac{17}{\cancel{2}} = c+4 \quad \text{—}$$

or, $$2g+17f = c+4 \quad \text{— (1)}$$

$$\cancel{2}g \times \frac{7}{\cancel{2}} + \cancel{2}f \times \frac{6}{\cancel{2}} = c+11$$

or, $$7g+6f = c+11 \quad \text{— (2)}$$

and $$\cancel{2}g \times -\frac{1}{\cancel{2}} + \cancel{2}f \times \frac{22}{\cancel{2}} = c+3$$

or, $$-g+22f = c+3 \quad \text{— (3)}$$

subtracting (1) from (2) we have,

$$5g-11f-7=0 \quad \text{— (4)}$$

and subtracting (3) from (2), we have

$$8g-16f=8$$

or, $$g-\frac{16f}{8}=\frac{8}{8}$$

or, $$g-2f=1 \quad \text{— (5)}$$

Solving (4) and (5), we have

$$5g-11f-7=0$$

$$g-2f-1=0$$

Solving by cross multification,

$$\frac{g}{-14-11} = \frac{f}{-7-5} = \frac{1}{-11-10}$$

$$\frac{g}{-3} = \frac{f}{-2} = -\frac{1}{1}$$

$\therefore g = -3$ and $f = -2$

~~From (3), we have~~ putting the values of g, f in eqn-(3), we have

$$c = -g + 22f - 3$$

$$c = 3 - 44 - 3$$

$$= -44$$

Putting the values of g, f and c in eqn. $x^2 + y^2 + 2gx + 2fy + c = 0$, we have

$$x^2 + y^2 - 6x - 4y - 44 = 0$$

required equation